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On ekpyrotic brane collisions

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Abstract

We derive the five-dimensional metrics which describe a non-singular boundary brane collision in the ekpyrotic scenario in the context of general relativity, taking into account brane tension. We show that the metrics constrain matter created in the collision to have negative energy density or pressure. In particular, the minimal field content of heterotic M-theory leads to negative energy density. We also consider bulk brane-boundary brane collisions and show that the collapse of the fifth dimension is an artifact of the four-dimensional effective theory.

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1 Introduction

A new cosmological framework called the ekpyrotic scenario has recently been under intense discussion [1-13]. The scenario has heterotic M-theory [14, 15] as its origin and brane cosmology [16-32] as its context. The setting for the ekpyrotic scenario is an 11-dimensional spacetime with the topology $M_{10} \times S_1/Z_2$, with boundary branes at the orbifold fixed points where spacetime terminates. The boundary branes are called the visible and the hidden brane, with the visible brane identified with our universe. Six dimensions are compactified on a Calabi-Yau threefold, leaving the theory effectively five-dimensional.

In the original proposal for a realisation of the ekpyrotic scenario [1, 4] a third brane travels from the hidden brane to collide with the visible brane in an event called ekpyrosis. Ekpyrosis is posited to transfer some of the energy of this bulk brane onto the visible brane and thus ignite the big bang at some finite temperature. A major problem was that during the journey of the bulk brane, the direction transverse to the branes was contracting, whereas stabilisation was considered necessary for the post-ekpyrosis era. According to [8], in order to reverse the contraction either the null energy condition has to be violated or the scale factor of the transverse direction has to pass through zero. The authors chose not to violate the null energy condition, and in the second proposal [8, 11] there is no bulk brane but the boundary branes themselves collide and then bounce apart, so that the scale factor passes through zero in what is hoped to be a non-singular process. This approach has also served as a vital ingredient in the so-called "cyclic model of the universe" [35, 36].

The analysis has been done in the context of a four-dimensional effective theory. The evolution of cosmological perturbations within this framework has been debated, with particular concern about the matching conditions across the bounce and the validity of the analysis near the singular point where the scale factor vanishes [5, 9, 10, 11, 12]. However, it is not obvious that even the homogeneous and isotropic background is correctly treated by the four-dimensional effective theory. It has been observed that the ansatz on which the four-dimensional effective theory is based cannot support brane matter created by ekpyrosis [6, 7] and does not satisfy the five-dimensional equations of motion [6], at least with the approximations made in [1]. There are also quite general concerns about the validity of four-dimensional effective theories involving integration over the transverse direction in brane cosmologies [29, 6].

The present paper consists of two main parts. After collecting some necessary equations in section 2, we study boundary brane collisions with the full five-dimensional equations in section 3. We derive the metrics possible under the assumption that the collision is non-singular, study which of these are ruled out by the field equations and see what are the constraints on brane matter created by boundary brane ekpyrosis. We compare with the approach of [8] and discuss ways to avoid the constraints on brane matter. In section 4 we study bulk brane-boundary brane collisions in the moduli space approximation using the five-dimensional equations. We reassess the collapse of the transverse direction and consider the validity of the moduli space approximation. In section 5 we summarise our results and comment on the implications for the "cyclic model of the universe".

2 The set-up

The action and the metric. The action for both the old and the new ekpyrotic scenario consists of three parts:

$$S = S_{het} + S_{BI} + S_{matter} \; , \tag{1}$$

where S_{het} is the action of five-dimensional heterotic M-theory with minimal field content, S_{BI} describes the brane interaction responsible for brane movement and S_{matter} describes brane matter created by ekpyrosis.

The simplified action of five-dimensional heterotic M-theory is [15, 16]

$$S_{het} = -\frac{M_5^3}{2}\int_{M_5} d^5x \sqrt{-g}\left[R - \frac{1}{2}\partial_A\phi\,\partial^A\phi - \frac{3}{2}\frac{1}{5!}e^{2\phi}F_{ABCDE}F^{ABCDE}\right]$$
$$-\sum_{i=1}^{3} 3\alpha_i M_5^3 \int_{M_4^{(i)}} d^4\sigma_{(i)}\left[\sqrt{-h_{(i)}}e^{-\phi} - \frac{1}{4!}A_{ABCD}\,\partial_\alpha X^A_{(i)}\partial_\beta X^B_{(i)}\partial_\gamma X^C_{(i)}\partial_\delta X^D_{(i)}\epsilon^{\alpha\beta\gamma\delta}\right] \; , \tag{2}$$

where M_5 is the Planck mass in five dimensions, R is the scalar curvature in five dimensions, e^{ϕ} is essentially the volume of the Calabi-Yau threefold and A_{ABCD} is a four-form gauge field with field strength $F = dA$. The Latin indices run from 0 to 4 and the Greek indices run from 0 to 3. The spacetime is a five-dimensional manifold $M_5 = M_4 \times S_1/Z_2$ with coordinates x^A. The four-dimensional manifolds $M_4^{(i)}$, $i = 1; 2; 3$, are the orbifold planes, called the visible, hidden and bulk branes respectively, with internal coordinates $\sigma_{(i)}$ and tensions $\alpha_i M_5^3$. The tensions are denoted $\alpha_1 = -\alpha$, $\alpha_2 = \alpha - \beta$ and $\alpha_3 = \beta$. We leave the sign of α undetermined; the tension of the bulk brane is always positive, $\beta > 0$, and we will assume $\beta < |\alpha|$. The tensor g_{AB} is the metric on M_5 and $h^{(i)}$ are the induced metrics on $M_4^{(i)}$. The functions $X^A_{(i)}(\sigma_{(i)})$ are the coordinates in M_5 of a point on $M_4^{(i)}$ with coordinates $\sigma_{(i)}$, in other words they give the embedding of the branes into spacetime.

The brane interaction term is due to non-perturbative M-theory effects [1]. In [1], the interaction was given in the context of a four-dimensional effective action, and it is not known what it looks like in the five-dimensional picture. However, since the string coupling is posited to vanish at the brane collision, the contribution of the brane interaction goes asymptotically to zero before the collision and rises from zero (or stays zero) after the collision. We will only need this crucial property for our analysis; the detailed form of the brane interaction will be unimportant.

Brane matter is assumed to be created in the brane collision. In the old ekpyrotic scenario, the collision took place between the bulk brane and the visible brane, so that the hidden brane remained empty. In the new scenario, the collision is between the boundary branes, so we allow for the possibility of matter creation on the hidden brane as well. The brane matter action is

$$S_{matter} = \sum_{i=1}^{2} \int_{M_4^{(i)}} d^4 \,_{(i)} \sqrt{\quad h_{(i)}}\, L_{matter(i)} \,. \tag{3}$$

We will consider the following metric ansatz ($t \quad x^0$, $y \quad x^4$):

$$ds^2 = \quad n(t;y)^2 dt^2 + a(t;y)^2 \sum_{j=1}^{3} (dx^j)^2 + b(t;y)^2 dy^2 \,. \tag{4}$$

The branes are taken to be flat and parallel, and we will not consider brane bending, so the embedding is

$$X^A_{(i)}(\,_{(i)}) = (t; x^1; x^2; x^3; y_i) \,; \tag{5}$$

with $y_1 = 0$, $y_2 = R$ and $y_3 = Y(t)$, where R is a constant.

The field equations. From the action (2) with the metric (4) we obtain the following field equations for A_{ABCD} and :

$$\frac{3}{5!} e^{2} F_{ABCDE} F^{ABCDE} + \sum_{i=1}^{2} (y \quad y_i) b^{\,1} 6 \,_i e \quad = 0$$

$$D_M (e^{2} F^{MABCD}) + \,^{[A}_{0} \,^{B}_{1} \,^{C}_{2} \,^{D]}_{3} \sum_{i=1}^{2} (y \quad y_i)(\quad g)^{1=2} 2 \,_i = 0 \,; \tag{6}$$

where D_M is the covariant derivative. The contribution of brane interaction terms which might couple to A_{ABCD} or and thus affect the equations of motion has been omitted.

The Einstein equation. The Einstein equation

$$G_{AB} = \frac{1}{M_5^3} T_{AB} \tag{7}$$

for the action (1) and the metric (4) reads in component form

$$
\begin{aligned}
G^t{}_t &= \frac{3}{b^2}\left(\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right)-\frac{3}{n^2}\left(\frac{\dot a}{a}\left(\frac{\dot a}{a}+\frac{\dot b}{b}\right)\right)\\
&= -\frac{1}{4}n^{-2}\dot\phi^2-\frac{1}{4}b^{-2}\phi'^2+\frac{3}{4}\frac{1}{5!}e^{2\phi}F_{ABCDE}F^{ABCDE}\\
&\quad -\frac{1}{M_5^3}\sum_{i=1}^{3}\delta(y-y_i)b^{-1}\rho_{b(i)}+\frac{1}{M_5^3}T^t{}_t(BI)\\
G^j{}_j &= \frac{1}{b^2}\left[2\frac{a''}{a}+\frac{n''}{n}+\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right)-\frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right)\right]\\
&\quad -\frac{1}{n^2}\left[2\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\dot a}{a}\left(\frac{\dot a}{a}-2\frac{\dot n}{n}\right)+\frac{\dot b}{b}\left(2\frac{\dot a}{a}-\frac{\dot n}{n}\right)\right]\\
&= -\frac{1}{4}n^{-2}\dot\phi^2-\frac{1}{4}b^{-2}\phi'^2+\frac{3}{4}\frac{1}{5!}e^{2\phi}F_{ABCDE}F^{ABCDE}\\
&\quad +\frac{1}{M_5^3}\sum_{i=1}^{3}\delta(y-y_i)b^{-1}p_{b(i)}+\frac{1}{M_5^3}T^j{}_j(BI)\\
G^y{}_y &= \frac{3}{b^2}\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right)-\frac{3}{n^2}\left(\frac{\ddot a}{a}+\frac{\dot a}{a}\left(\frac{\dot a}{a}-\frac{\dot n}{n}\right)\right)\\
&= -\frac{1}{4}n^{-2}\dot\phi^2+\frac{1}{4}b^{-2}\phi'^2+\frac{3}{4}\frac{1}{5!}e^{2\phi}F_{ABCDE}F^{ABCDE}+\frac{1}{M_5^3}T^y{}_y(BI)\\
G_{ty} &= 3\left(\frac{n'}{n}\frac{\dot a}{a}+\frac{a'}{a}\frac{\dot b}{b}-\frac{\dot a'}{a}\right)=\frac{1}{2}\dot\phi\phi'+\frac{1}{M_5^3}T_{ty}(BI)\ ;
\end{aligned}
\tag{8}
$$

where dots and primes stand for derivatives with respect to t and y, respectively, $T_{AB}(BI)$ represents the brane interaction and $\rho_{b(i)}$ and $p_{b(i)}$ are the energy density and pressure of brane i:

$$
\begin{aligned}
\rho_{b(i)} &= \rho_{m(i)}+3M_5^3\alpha_i e^{-\phi}\\
p_{b(i)} &= p_{m(i)}-3M_5^3\alpha_i e^{-\phi}\ .
\end{aligned}
\tag{9}
$$

The terms $\rho_{m(i)}$ and $p_{m(i)}$ are the contribution of brane matter, present only after ekpyrosis. Note that under the assumption of homogeneity and isotropy, the energy-momentum tensor of brane matter necessarily has the ideal fluid form. The delta function part of (8) reads [19]

$$
\begin{aligned}
3\frac{1}{b}\frac{a''}{a} &= -\frac{1}{M_5^3}\sum_{i=1}^{2}\delta(y-y_i)\rho_{b(i)}+O(t)\\
\frac{1}{b}\left(2\frac{a''}{a}+\frac{n''}{n}\right) &= \frac{1}{M_5^3}\sum_{i=1}^{2}\delta(y-y_i)p_{b(i)}+O(t)\ ;
\end{aligned}
\tag{10}
$$

where $O(t)$ stands for possible terms due to a delta function part in the energy-momentum tensor of the brane interaction. We assume here and in what follows that terms due to the brane interaction vanish at least as fast as t

where j and $j' \neq j$ are spatial directions parallel to the brane. The Riemann tensor on the brane is[1]

$$
\begin{aligned}
{}^{(i)}R_{\hat\tau\hat\jmath\hat\tau\hat\jmath} &= -\frac{1}{n^2}\left(\frac{\ddot a}{a} - \frac{\dot a}{a}\frac{\dot n}{n}\right)\Bigg|_{y=y_i} \\
{}^{(i)}R_{\hat\jmath\hat\jmath'\hat\jmath\hat\jmath'} &= \frac{1}{n^2}\frac{\dot a^2}{a^2}\Bigg|_{y=y_i} ;
\end{aligned}
\tag{19}
$$

where the index i refers to the four-dimensional quantity measured on brane i. Concentrating on the visible brane, we have

$$
\begin{aligned}
{}^{(1)}R_{\hat\tau\hat\jmath\hat\tau\hat\jmath} &= -\frac{\ddot a}{a}\Bigg|_{y=0} \\
{}^{(1)}R_{\hat\jmath\hat\jmath'\hat\jmath\hat\jmath'} &= \frac{\dot a^2}{a^2}\Bigg|_{y=0} .
\end{aligned}
\tag{20}
$$

The boundedness requirement. Let us first consider the boundedness of the Riemann tensor on the brane. From (20) we see that if the scale factor on the visible brane approaches zero or diverges, the Riemann tensor on the brane grows without bound[2]. We conclude that the scale factor on the visible brane approaches a finite value as the branes approach each other.

Let us now turn to the Riemann tensor in the bulk. Since the calculation is the same before and after the collision, we temporarily drop the index $(\pm)$. We will denote the first terms in the series expansion (12) of n and a whose y–derivative does not vanish everywhere by n_ℓ and $a_{\bar m}$. (That is, $n_i'(y) = 0\ \forall\, i < \ell$, $a_i'(y) = 0\ \forall\, i < \bar m$.) With the series expansion (12), the leading terms of the Riemann tensor (14)–(18) read

$$
R_{\hat\tau\hat\jmath\hat\tau\hat\jmath} \simeq -t^{2k+\ell-l+\bar m-m}\frac{1}{b_k^2}\frac{a'_{\bar m}}{a_m}\frac{n'_\ell}{n_l} - t^{-2l-2}\frac{1}{n_l^2}m(m-l-1) \tag{21}
$$

$$
R_{\hat\jmath\hat\jmath'\hat\jmath\hat\jmath'} \simeq -t^{2k+2\bar m-2m}\frac{1}{b_k^2}\frac{a'^2_{\bar m}}{a_m^2} + t^{-2l-2}\frac{1}{n_l^2}m^2 \tag{22}
$$

$$
R_{\hat\tau\hat\jmath\hat y\hat\jmath} \simeq t^{-k-l-1}\frac{1}{n_l b_k}\left(-t^{\ell-l}m\frac{n'_\ell}{n_l} + t^{\bar m-m}(k-\bar m)\frac{a'_{\bar m}}{a_m}\right) \tag{23}
$$

$$
R_{\hat\tau\hat y\hat\tau\hat y} \simeq t^{2k+\ell-l}\frac{1}{b_k^2}\left(\frac{n''_\ell}{n_l} - \frac{b'_k}{b_k}\frac{n'_\ell}{n_l}\right) - t^{-2l-2}\frac{1}{n_l^2}k(k-l-1) \tag{24}
$$

$$
R_{\hat\jmath\hat y\hat\jmath\hat y} \simeq -t^{2k+\bar m-m}\frac{1}{b_k^2}\left(\frac{a''_{\bar m}}{a_m} - \frac{b'_k}{b_k}\frac{a'_{\bar m}}{a_m}\right) + t^{-2l-2}\frac{1}{n_l^2}mk . \tag{25}
$$

[1] Note that this is not the same as the five-dimensional bulk Riemann tensor evaluated at the brane position.

[2] The same conclusion can also be obtained from the Riemann tensor in the bulk with the help of the junction conditions (11), assuming that the brane energy density and pressure remain bounded.

Recall that we have $n(t;0) = 1$ by choice of coordinates, and that according to (20) $a(t;0) \simeq$ finite constant. This is only possible if $l, m \le 0$ and $n_i(0) = 0 \ \forall\, i \neq 0$, $a_i(0) = 0 \ \forall\, i < 0$. Note that the coefficients only have to vanish at the visible brane; it is possible for them to be non-zero elsewhere. In particular, if $l < 0$, then n_l must be zero at the visible brane, but not everywhere. This of course means that n_l' does not vanish identically, so that $\tilde{l} = l$. Similarly, $m < 0$ implies $\tilde{m} = m$. Also, finiteness of the brane energy density and pressure imply, via the junction conditions (11)

$$
\begin{aligned}
n_i'(0) &= a_i'(0) = 0 \qquad \forall\, i < k &(26)\\
n_k'(0) &\neq 0\,; \ a_k'(0) \neq 0\,; &(27)
\end{aligned}
$$

which obviously means $\tilde{l}, \tilde{m} \le k$. Note that since l and m have turned out to be integers, k must also be an integer.

Let us assume that $l < 0$. Then the first term of (21) is proportional to $|t|^{-2k+\tilde{m}-m} \gtrsim |t|^{-k}$ with a non-vanishing coefficient, and thus divergent. Since the second term is proportional to at worst $|t|^{-2l-2} = |t|^{2|l|-2} \lesssim |t|^0$, it is bounded and cannot cancel the divergence of the first term. So, we must have $l = 0$.

Let us now assume that $m < 0$. Then, in order for it to be possible for the divergent terms in (21) and (22) to cancel, we must have $k = 1$ and $\tilde{l} = 0$. But then the first term in (24) is more divergent than the second, and thus its coefficient must vanish, yielding $n_0'/b_k =$ constant. However, according to (26) we must have $n_0'(0) = 0$, implying that n_0' vanishes everywhere, in contradiction with the definition of $\tilde{l}$. We conclude that $m = 0$.

Given $l = m = 0$, it follows straightforwardly from (21), (22) and (23) that $\tilde{l} = \tilde{m} = k$. The first three components of the Riemann tensor provide no further insight. The results of the remaining two equations depend on the value of k, so let us consider the different possibilities separately.

$k = 1$. For the simplest possibility, the cancellation of the divergences in (24) and (25) is equivalent to the following equations

$$
\begin{aligned}
\frac{1}{b_k^2}\left(n_1'' - \frac{b_k'}{b_k} n_1'\right) - n_1 &= 0\\
\frac{1}{b_k^2}\left(a_1'' - \frac{b_k'}{b_k} a_1'\right) - a_1 &= 0\,. &(28)
\end{aligned}
$$

With the coordinate choice $b_k(y) = B$, with B a positive constant, the above equations reduce to

$$
\begin{aligned}
n_1'' - B^2 n_1 &= 0\\
a_1'' - B^2 a_1 &= 0\,; &(29)
\end{aligned}
$$

with the solutions

$$
\begin{aligned}
n_1(y) &= N_1 \sinh(By) \\
a_1(y) &= A_1 \sinh(By) + \tilde{A}_1 \cosh(By) \; ; \qquad (30)
\end{aligned}
$$

where N_1 and A_1 are non-zero constants, $\tilde{A}_1$ is a constant which may be zero and we have taken into account $n_1(0) = 0$. The metric (4) near the collision is

$$
\begin{aligned}
b(t;y) &= Bt + O(t^2) \\
n(t;y) &= 1 + N_1 \sinh(By)t + O(t^2) \\
a(t;y) &= 1 + \left[A_1 \sinh(By) + \tilde{A}_1 \cosh(By)\right] t + O(t^2) \; ; \qquad (31)
\end{aligned}
$$

where we have set $A_0 = 1$.

$k \geq 2$. Now the cancellation of the leading divergences in (24) and (25) is equivalent to the equations

$$
\begin{aligned}
\frac{1}{b_k^2}\left(n_k'' - \frac{b_k'}{b_k} n_k'\right) - 2\delta_{2k} &= 0 \\
\frac{1}{b_k^2}\left(a_k'' - \frac{b_k'}{b_k} a_k'\right) &= 0 \; ; \qquad (32)
\end{aligned}
$$

where δ_{2k} is the Kronecker delta. The above equations have the solution

$$
\begin{aligned}
\frac{1}{b_k} n_k' &= N_k + 2\delta_{2k} \int_0^y dz\, b_k(z) \\
\frac{1}{b_k} a_k' &= A_k \; ; \qquad (33)
\end{aligned}
$$

where N_k and A_k are non-zero constants. With the metric choice $b_k(y) = B$ the solutions reduce to

$$
\begin{aligned}
n_k(y) &= N_k By + \delta_{2k} B^2 y^2 \\
a_k(y) &= A_k By + \tilde{A}_k \; ; \qquad (34)
\end{aligned}
$$

where $\tilde{A}_k$ is a constant.

The cancellation of subleading divergences in (24) and (25) imposes $k-1$ relations between the higher order coefficients n_{k+i} and a_{k+i}. To leading order, the metric (4) is

$$
\begin{aligned}
b(t;y) &= Bt^k + O(t^{k+1}) \\
n(t;y) &= 1 + (N_k By + \delta_{2k} B^2 y^2) t^k + O(t^{k+1}) \\
a(t;y) &= 1 + \sum_{i=1}^{k-1} A_i t^i + (A_k By + \tilde{A}_k) t^k + O(t^{k+1}) \; ; \qquad (35)
\end{aligned}
$$

where A_i are constants which may be zero, and we have set $A_0 = 1$.

The no- ow requirement. In addition to the boundedness requirement, there is another constraint the metric should satisfy: energy should not owo spacetime at the branes,

$$G_{\hat{t}\hat{y}}\Big|_{y=y_i} = 0 \; : \tag{36}$$

The condition (36) is satis ed to leading order by virtue of the component (16) of the Riemann tensor being bounded. Subleading terms of (36) involve higher order coe cients n_{k+i} and a_{k+i}. As noted, the boundedness of the components (17) and (18) of the Riemann tensor imposes $k \;\; 1$ conditions on the same coe cients. However, the boundedness and no- ow conditions are compatible and can all be simultaneously satis ed.

3.3 The energy-momentum tensor

We have derived the metrics allowed by the non-singularity and non-boundedness conditions of the Riemann tensor. Let us now see which of these metrics, (31) and (35), are allowed by the same conditions of the energy-momentum tensor.

In the local orthonormal basis, the bulk energy-momentum tensor given in (8) is (after eliminating A_{ABCD} by using its equation of motion)

$$\frac{1}{M_5^3}T_{\hat{t}\hat{t}} = \frac{1}{4}n^{\;\;2}\,\_^{2} + \frac{1}{4}b^{\;2}\;{}^{02} + \frac{3}{4}\;{}^{2}e^{\;\;2} \;+\; \frac{1}{M_5^3}T_{\hat{t}\hat{t}}(BI) \tag{37}$$

$$\frac{1}{M_5^3}T_{\hat{\jmath}\hat{\jmath}} = \frac{1}{4}n^{\;\;2}\,\_^{2} \quad \frac{1}{4}b^{\;2}\;{}^{02} \quad \frac{3}{4}\;{}^{2}e^{\;\;2} \;+\; \frac{1}{M_5^3}T_{\hat{\jmath}\hat{\jmath}}(BI) \tag{38}$$

$$\frac{1}{M_5^3}T_{\hat{y}\hat{y}} = \frac{1}{4}n^{\;\;2}\,\_^{2} + \frac{1}{4}b^{\;2}\;{}^{02} \quad \frac{3}{4}\;{}^{2}e^{\;\;2} \;+\; \frac{1}{M_5^3}T_{\hat{y}\hat{y}}(BI) \tag{39}$$

$$\frac{1}{M_5^3}T_{\hat{t}\hat{y}} = \frac{1}{2}n^{\;\;1}b^{\;1}\,\_\;{}^{0} + \frac{1}{M_5^3}T_{\hat{t}\hat{y}}(BI) \; ; \tag{40}$$

and the brane energy-momentum tensor given in (8) and (9) is

$$\begin{aligned} {}^{(i)}T_{\hat{t}\hat{t}} &= \quad{}_{m(i)} + 3M_5^3 \;{}_{i}e \quad + {}^{(i)}T_{\hat{t}\hat{t}}(BI) \\ {}^{(i)}T_{\hat{\jmath}\hat{\jmath}} &= p_{m(i)} \quad 3M_5^3 \;{}_{i}e \quad + {}^{(i)}T_{\hat{\jmath}\hat{\jmath}}(BI) \; ; \end{aligned} \tag{41}$$

where the index i refers to the four-dimensional quantity measured on brane i. Since there is no bulk brane, ${}_{i} = (\;\; 1)^{i}$.

The boundedness requirement. Requiring the bulk energy-momentum tensor to remain bounded gives the conditions

$$
\begin{aligned}
n^{\;1}\,\_ &= O(t^{0}) & (42)\\
b^{\;1\;\;0} &= O(t^{0}) & (43)\\
e &= O(t^{0})\;: & (44)
\end{aligned}
$$

The boundedness of the brane energy-momentum tensor does not impose any additional constraints. In terms of the series expressions (12) and (13), the conditions (42){(44) read, given l = 0,

$$
\begin{aligned}
{}_{i}(y) &= 0 \qquad 8\; i < 0 & (45)\\
{}^{0}_{i}(y) &= 0 \qquad 8\; i < k & (46)\\
v_{0}(y) &\; 6=\; 0\;: & (47)
\end{aligned}
$$

The no- ow requirement. In addition to the boundedness requirement, we should again require that energy does not ow away from spacetime at the branes,

$$T_{ty}\Big|_{y=y_i} = 0\;: \tag{48}$$

With (40) this gives, since $b^{\;1\;\;0}$ is nite at the branes due to the equation of motion (52),

$$n^{\;1}\,\_\Big|_{y=y_i} \overset{t!\;0}{!}\; 0\;; \tag{49}$$

a slightly but crucially stronger condition than (42). In terms of the series expression (13), the condition (49) reads

$${}_{1}(y_i) = 0\;: \tag{50}$$

Newton's constants. There has been some concern [8, 36] that a vanishing transverse direction leads to a divergent Newton's constant and therefore large quantum uctuations. However, the calculations have dealt with a Newton's constant in a four-dimensional e ective theory. The Newton's constant measured in the bulk of the ve-dimensional theory is of course constant, while the Newton's constant measured on a brane is ${}_{i}e\;=(16\,M_{5}^{3})$, where ${}_{i}$ is the brane tension and e is evaluated at the brane position [29, 7]. As long as the size of the Calabi-Yau threefold stays nite the Newton's constant(s) in the ve-dimensional theory are completely well-behaved, regardless of the behaviour of the transverse direction.

The eld equation. There is one more condition that the energy-momentum tensor should satisfy: covariant conservation. In the case of the metric, covariant conservation (of the Einstein tensor) is an identity, but for the energy-momentum tensor it provides a non-trivial constraint. The covariant conservation law of the energy-momentum tensor is in this case (after eliminating A_{ABCD} by using its equation of motion) equivalent to the equation of motion of ,

$$n^{2} \quad + \quad \left(\frac{n}{n} + 3\frac{a}{a} + \frac{b}{b} \right)$$

$$+ b^{2} \quad {}^{00} + \quad \frac{n^{0}}{n} + 3\frac{a^{0}}{a} \quad \frac{b^{0}}{b} \quad {}^{0} + 3 \quad {}^{2} e^{2} \quad = \quad O(t^{k+1}) \tag{51}$$

$$(y \quad y_i)(b^{1} \; {}^{0} \quad 3 e \quad) \quad = \quad O(t) \; ; \tag{52}$$

where the right-hand sides are the possible contribution of the brane interaction. The brane part (52) of the eld equation is satis ed provided that

$$\frac{1}{b_k(y_i)} \; {}^{0}_{k}(y_i) \quad = \quad 3 \; e \quad {}^{0} \; : \tag{53}$$

Inserting the expansions (12) and (13) into the bulk part (51) of the eld equation and taking into account (42){(44) and the previous section's results $= m = 0$, $\tilde{} = \tilde{m} = k$, the leading terms are

$$kt^{1} \; {}_{1} + b_k^{2} t^{k} \quad {}^{00}_{k} \quad \frac{b^{0}_{k}}{b_k} \; {}^{0}_{k} \quad = \quad O(t^{k+1}) \; : \tag{54}$$

Let us consider di erent values of k separately.

$k = 1$. With the coordinate choice $b_k(y) = B$, (54) simpli es to

$${}^{00}_{1} \quad B^{2} \; {}_{1} \quad = \quad 0 \; ; \tag{55}$$

with the familiar solution $_1(y) = \cosh(By) + \tilde{} \sinh(By)$, where $; \tilde{}$ are constants. The requirement that no energy ows away from spacetime, (50) leads to $_1(y) = 0$. But according to (53), we should have ${}^{0}_{k}(y_i) \neq 0$. We conclude that $k = 1$ is ruled out by the no- ow condition of .

$k \quad 2$. In this case the leading term of (54) gives

$${}^{00}_{k} \quad \frac{b^{0}_{k}}{b_k} \; {}^{0}_{k} \quad = \quad 0 \; ; \tag{56}$$

the solution of which is also familiar,

$$
\begin{aligned}
\frac{1}{b_k}\,{}^{0}_{k} &= \text{constant} \\
&= 3\,e\ \ {}^{0}\,; \qquad (57)
\end{aligned}
$$

where we have on the second line used (53). With the coordinate choice $b_k(y) = B$ we have

$$
{}_k(y) = 3\,e\ \ {}^{0}By + '_k\,; \qquad (58)
$$

where $'_k$ is a constant. The subleading terms may involve the brane interaction and thus cannot provide any information.

3.4 Constraints on brane matter

We have derived the metrics allowed by the boundedness and no-cross conditions of the Riemann tensor, (31) and (35). We have then seen that the same conditions of the minimal energy-momentum tensor of heterotic M-theory metric allow only the metric (35). Let us now consider what the metrics (31) and (35) have to say on the issue of brane matter, setting for a moment aside the constraint due to the energy-momentum tensor.

It is a known feature of brane cosmologies that limitations on brane matter may arise in constrained metric configurations, most notably those with a factorisable metric [24, 25] or $b = 0$ [26, 27, 31]. The near-collision metrics (31) and (35) do not fall into either class, but they do have quite a restrictive form. Since we need to discuss the pre- and post-ekpyrosis eras separately, we return the index ( ).

k = 1. Putting together the junction conditions (11), the constraints (46) and (47) on , and the metric (31), we have

$$
\begin{aligned}
& N_1^{(\ )}\cosh(B^{(\ )}y_i) \\
&= \frac{1}{2}e\ \ {}^{0}\ \ \ (t)\frac{1}{6M_5^3}(\ \ 1)^i\ \ 2\ {}_{m(i)}(0) + 3p_{m(i)}(0) \qquad (59) \\
& A_1^{(\ )}\cosh(B^{(\ )}y_i) + A_1^{(\ )}\sinh(B^{(\ )}y_i) \\
&= \frac{1}{2}e\ \ {}^{0} +\ \ (t)\frac{1}{6M_5^3}(\ \ 1)^i\ {}_{m(i)}(0)\,; \qquad (60)
\end{aligned}
$$

where (t) is the step function, ${}_{m(i)}(0)$ ${}_{m(i)}(t = 0)$ is the energy density of matter created on brane i by ekpyrosis, and $p_{m(i)}(0)$ is the corresponding pressure. In general, we of course cannot set the two functions $b_{k(\ )}^{(\ )}(y)$ to a constant simultaneously both before and after ekpyrosis, so it should be understood that we are using different y–coordinate for the pre- and post-ekpyrosis eras.

For the pre-ekpyrosis era, (59) cannot be satisfied since the l.h.s. is different at different branes whereas ${}_0$ is constant due to the boundedness requirement (46). We conclude that $k_{(\ )} = 1$ is excluded.

For the post-ekpyrosis era, the junction conditions (59) and (60) read

$$
\begin{aligned}
6M_5^3 N_1^{(+)} &= 3\ M\ {}_5^3 e^{\ 0} + 2\ {}_{m(1)}(0) + 3\,p_{m(1)}(0) \\
6M_5^3 N_1^{(+)}\ \cosh(B^{(+)}\ R) &= 3\ M\ {}_5^3 e^{\ 0}\quad 2\ {}_{m(2)}(0)\quad 3p_{m(2)}(0) \\
6M_5^3 A_1^{(+)} &= 3\ M\ {}_5^3 e^{\ 0}\quad {}_{m(1)}(0) \\
6M_5^3\ \ A_1^{(+)}\ \cosh(B^{(+)}\ R) + \tilde{A}_1^{(+)}\ \sinh(B^{(+)}\ R) &= 3\ M\ {}_5^3 e^{\ 0} + \ {}_{m(2)}(0)\ ;
\end{aligned}
\tag{61}
$$

The last two equations simply give A_1 and $\tilde{A}_1$ in terms of ${}_{m(1)}(0)$ and ${}_{m(2)}(0)$. However, the first two equations provide the constraint

$$
\frac{3\ M\ {}_5^3 e^{\ 0}\quad 2\ {}_{m(2)}(0)\quad 3p_{m(2)}(0)}{3\ M\ {}_5^3 e^{\ 0} + 2\ {}_{m(1)}(0) + 3\,p_{m(1)}(0)} = \cosh(B^{(+)}\ R) > 1\ ;
\tag{62}
$$

which implies

$$
2\ {}_{m(1)}(0) + 3\,p_{m(1)}(0) + 2\ {}_{m(2)}(0) + 3\,p_{m(2)}(0) < 0\ :
\tag{63}
$$

If we want to avoid negative energy densities we will inevitably have negative pressures. Having matter with positive energy density on the negative tension brane may be problematic, since the Newton's constant on a brane is proportional to the brane tension [29, 7]. For the standard four-dimensional Hubble law $H^2 = 8\ G_N\ {}_m/3$, it would be impossible for $G_N\ {}_m$ to be negative, but in the ekpyrotic scenario it may be possible, depending on the exact form of the Hubble law on the brane [7]. In the simple case that the hidden brane is empty, matter on the visible brane must satisfy $p_{m(1)}(0) <\ \ 2/3\ {}_{m(1)}(0)$. In any case, matter on at least one brane will not be just radiation but something more exotic.

Note that (62) relates the post-ekpyrosis expansion velocity $B^{(+)}\ R$ directly to the brane energy densities and pressures, and thus to the ekpyrotic temperature. For ekpyrotic temperatures much smaller than the scale of the brane tension (which is reasonably of the order of the Planck scale), the velocity $B^{(+)}\ R$ will be small, of the order of $T^2/(j\ \ jM_5^3 e^{\ 0})^{1/2}\quad T^2 M_4/M_5^3$, where T is the ekpyrotic temperature and M_4 is the Planck mass on the visible brane, and we have taken into account the relation $j\ \ je^{\ 0} = 16\ M\ {}_5^3/M_4^2$ [29, 7].

$k \quad 2$. Putting together the junction conditions (11), the constraints (46) and (47) on and the metric (35), we have

$$A_{k_{(\ )}} = \frac{1}{2M_5^3}\, e \quad {}^{0} + \quad (t)\frac{1}{6M_5^3}(\quad 1)^i \quad {}_{m(i)}(0)$$
$$N_{k_{(\ )}} + 2 \quad {}_{2k_{(\ )}} \int_0^{y_i} dz b_{k_{(\ )}}^{(\ )}(z) = \frac{1}{2M_5^3}\, e \quad {}^{0} \quad (t)\frac{1}{6M_5^3}(\quad 1)^i \quad 2 \quad {}_{m(i)}(0) + 3\, p_{m(i)}(0) \qquad (64)$$

For the pre-ekpyrosis era we just obtain the requirement $k_{(\ )} \quad 3$. For the post-ekpyrosis era we obtain the following constraints on brane matter

$$\begin{aligned} {}_{m(1)}(0) &= \quad {}_{m(2)}(0) \\ p_{m(1)}(0) &= \quad p_{m(2)}(0) \quad 4M_5^3 \quad {}_{2k_{(+)}} \int_0^R dz b_{k_{(+)}}^{(+)}(z) : \end{aligned} \qquad (65)$$

If the energy density and pressure of matter created on the visible brane is positive, a corresponding negative energy density, along with negative pressure has to be created on the hidden brane. So, matter on at least one of the branes violates the null energy condition[3]. We conclude that $k_{(+)} \quad 2$ is excluded by the null energy condition.

3.5 Comparison with the Milne metric

A preliminary investigation into how a brane collision which looks singular in a four-dimensional effective theory might be well-behaved in five dimensions was conducted in [8]. It was assumed that near the collision one can neglect the tension of the branes and approximate the five-dimensional spacetime with a compactified Minkowski metric. (A part of) the Minkowski spacetime can be written in Milne coordinates, so that it looks as follows:

$$ds^2 = \quad dt^2 + \sum_{j=1}^{3} (dx^j)^2 + B^2 t^2 dy^2 ; \qquad (66)$$

where B is a positive constant. This approximation essentially consists of the following assumptions: near the collision i) the time-dependence of the metric coefficient b is linear, ii) the time-dependence of the metric coefficients n and a is of higher order than that of b and iii) the y–dependence in the metric coefficients can be neglected.

We have now studied the brane collision with the five-dimensional equations, under the assumption that the five-dimensional theory is non-singular. The metric near a non-

[3] Except in the trivial case $\ {}_{m(i)}(0) + p_{m(i)}(0) = 0$, possible for $k_{(+)} \quad 3$.

singular collision is, according to (31) and (35),

$$
\begin{aligned}
ds^2 \simeq & -(1 + n_{k(-)}(y)t^{k(-)})^2 dt^2 + \Big(1 + \sum_{i=1}^{k(-)-1} A_i t^i + a_{k(-)}(y)t^{k(-)}\Big)^2 \sum_{j=1}^{3} (dx^j)^2 \\
& + B^{(-)2} t^{2k(-)} dy^2 \, ; \qquad (67)
\end{aligned}
$$

where we have set $b^{(-)}_{k(-)} = B^{(-)}$, and $n_{k(-)}$ and $a_{k(-)}$ are given in (31) and (35), and $k(-) \geq 3$, $k(+) \geq 1$. The constraints on brane matter (64) further say that in order to avoid negative energy densities we should have $k(+) = 1$. The above metric shows that before the collision b vanishes at least as fast as t^3, while after the collision it can vanish like t or like some higher power, that the time-dependence of n and a is at least of the same order as that of b, and that the y–dependence of n and a cannot be neglected. One can confirm from the Riemann tensor (14)-(18) that the t– and y–dependence of n and a do make a significant contribution to physical quantities arbitrarily near the collision.

The physical reason for the failure of the approximation (66) is the presence of brane tension (and brane matter). The Milne metric (66) describes a spacetime with no curvature, but the calculation leading to the true metric (67) shows that energy density on the brane will always curve spacetime in a manner that cannot be ignored; this is quite transparent in (59), (60) and (64), which show that $n_{k(-)}$ and (up to an additive constant) $a_{k(-)}$ are proportional to brane energy density and pressure. Were the brane tension turned off, the Milne metric (66) could be a good approximation.

3.6 Discussion on boundary brane collisions

Summary. We have derived the metrics which are possible under the assumption that the five-dimensional description remains valid, there is no curvature singularity and the brane energy density remains finite. We have then shown which of these metrics are allowed by the non-singularity and no-flow conditions of the energy-momentum tensor, and what are the constraints on brane matter. For the pre-ekpyrosis era everything works out, provided that the transverse direction vanishes at least as fast as t^3. However, for the post-ekpyrosis era, the single possibility that would avoid negative energy densities, the transverse direction vanishing like t, is excluded by the no-flow condition of the energy-momentum tensor.

Ways out. The rather forbidding conclusions on negative energy densities and/or pressures have been obtained in the context of five-dimensional heterotic M-theory with minimal field content and with dynamics dictated by general relativity and classical field theory. The relevant question is now which way the investigation should be generalised in order to avoid the unwanted results.

A simple remedy might be to turn on a new field in the five-dimensional action. The problem of negative energy density is solved if the energy flow associated with this new field compensates for the energy flow of at the boundary of spacetime, so that the no-flow condition does not imply $\bar{\ } = 0$ at the branes and thus exclude $k_{(+)} = 1$. However, the scenario would still be left with the problem of converting the negative pressure brane matter into positive pressure radiation and dealing with the inflation possibly onset by the negative pressure.[4] It would be desirable to avoid the exotic matter altogether.

One possibility is to consider string and quantum corrections to the five-dimensional action. One would expect string effects to play a role as the branes come near each other and quantum effects to become important near a curvature singularity. The problem of curing an ill-behaved collapse with string and quantum corrections (including the question of matching conditions) in the ekpyrotic scenario is in some ways reminiscent of the graceful exit problem in the pre-big bang model [37, 38, 39, 40]. In the ekpyrotic senario the problem may seem more tractable, since one is approaching the weak string coupling regime rather than the strong coupling regime as in the pre-big bang model, as emphasised in [8].

However, the problems of the ekpyrotic scenario and the pre-big bang model are of different nature. In the pre-big bang case, the curvature singularity appears from the equations of motion, so that it can in principle be avoided by changing the action. In the ekpyrotic case, the conclusions on negative energy density were made directly from the requirement of non-singularity without recourse to the equations of motion apart from the junction conditions. So, higher order curvature terms or string corrections to the five-dimensional action can only help by changing the junction conditions. Since the conclusions on negative energy density and pressure have been drawn from singular contributions and the string coupling is posited to vanish at the collision, string effects seem an unlikely remedy. Higher order curvature terms do in general change the junction conditions, but the survival of some constraints on brane matter seems likely. This is simply because though the near-collision metrics (31), (35) include enough free parameters to account for the four physical quantities (the energy densities and pressures on the two branes), the parameters enter in a quite restricted manner.

There is always the possibility of going further with the dimensional lifting, straight to the full eleven-dimensional string theory instead of the effective five-dimensional field theory. However, five-dimensional brane cosmologies have the merit of being relatively tractable and well-studied. In particular, the treatment of perturbations has been under study [33, 34], and may be applied to the five-dimensional picture of the ekpyrotic scenario.

[4] When $\bar{\ } \neq 0$ at the visible brane, the contribution of $\bar{\ }$ may decelerate the universe so much that even a large negative pressure does not lead to inflation. For details on the effects of $\bar{\ }$ on cosmology on the visible brane, see [7].

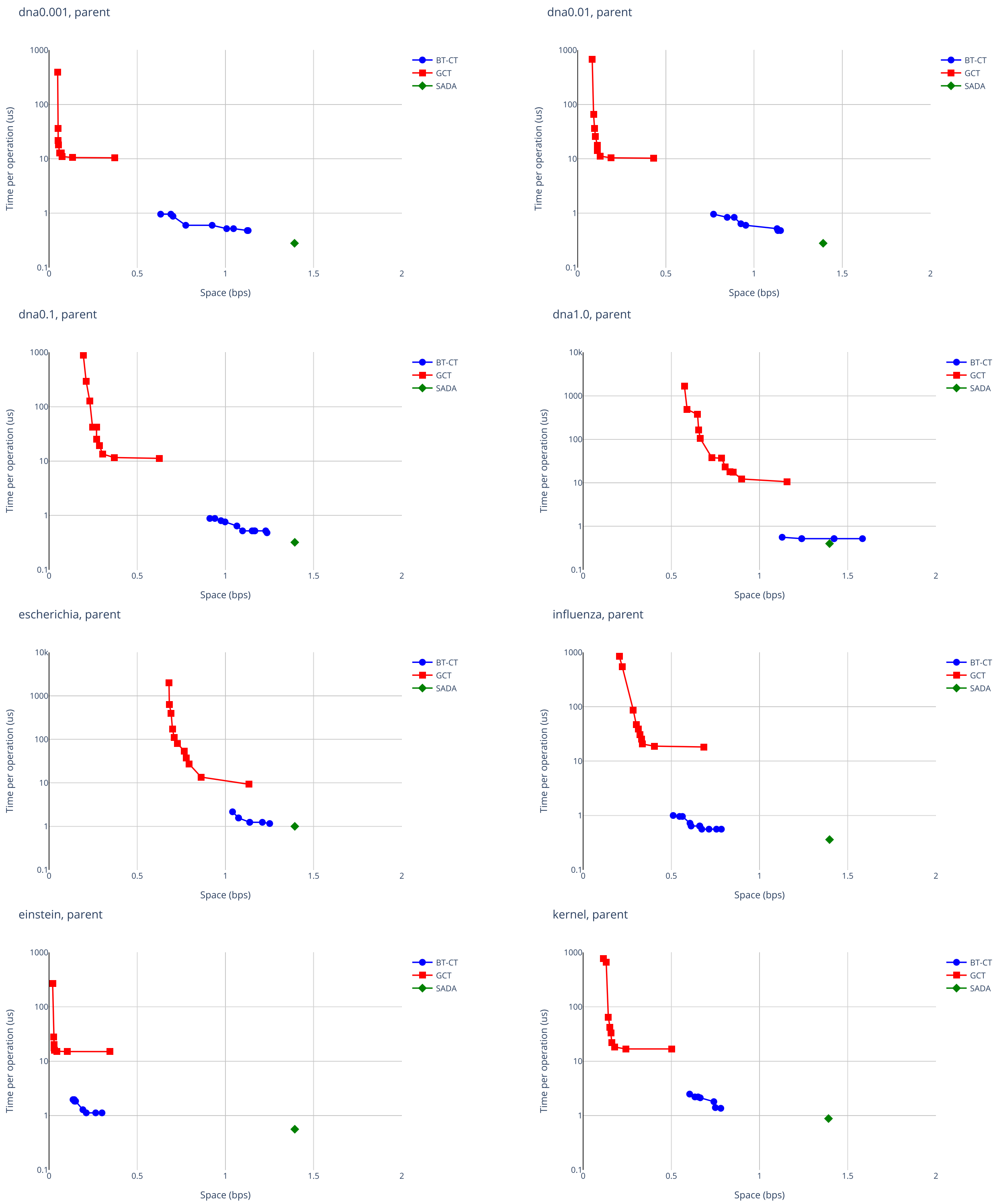

Fig. 10. Performance of parent in different BP representations. The y-axis is time in microseconds in log-scale.

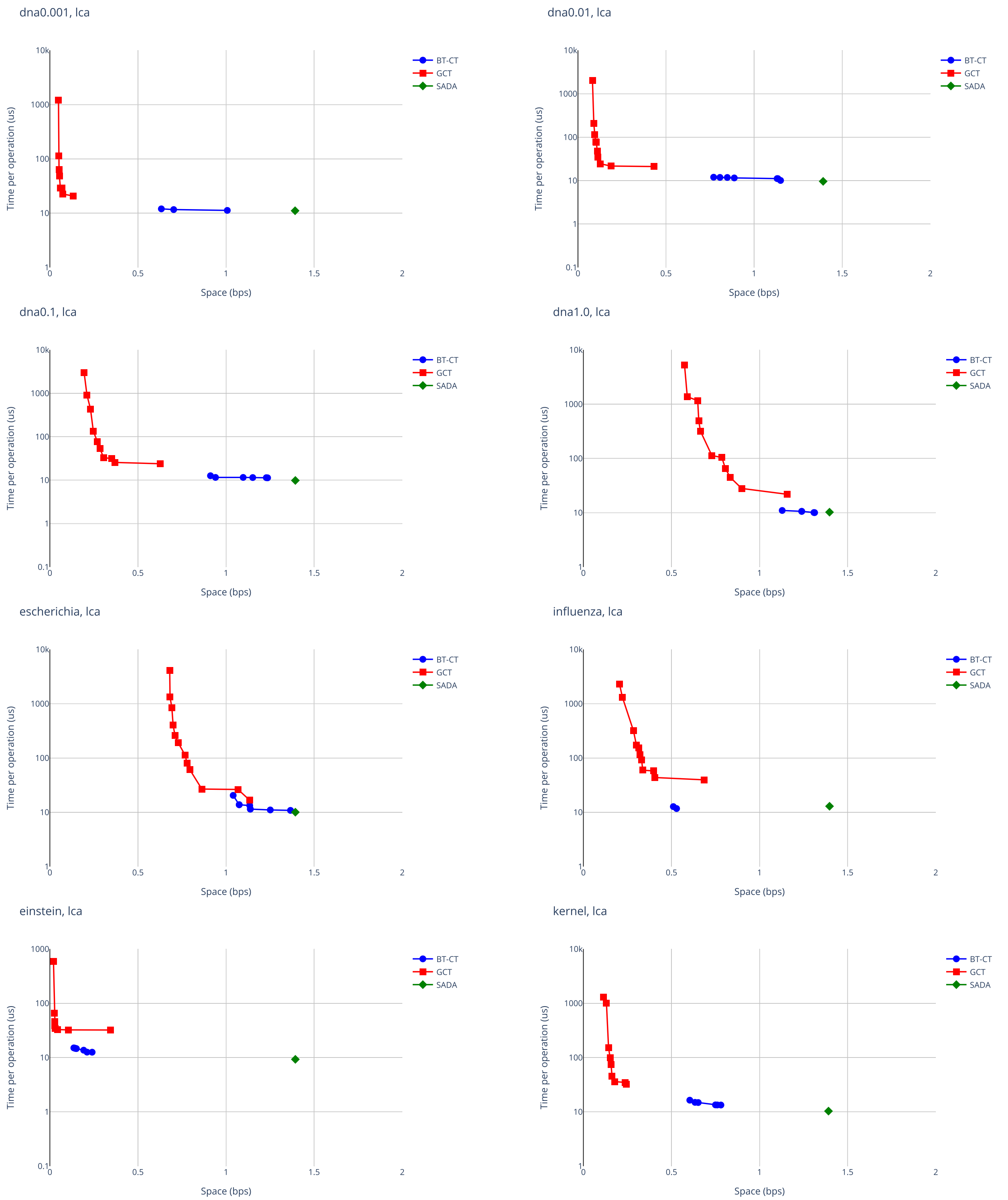

Fig. 12. Performance of lca in different BP representations. The y-axis is time in microseconds in log-scale.